Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 5, 1999

SINGULAR PERTURBATION IN BENDING PROBLEMS FOR ORTHOTROPIC PLATES

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UDC 539.3

The paper deals with a number of singularly perturbed boundary-value problems and variational inequalities that arise in the theory of bending of orthotropic plates with strong anisotropy of elastic properties.

Unlike in the previously studied boundary-value problems of plane elastic theory [1], asymptotic analysis of the boundary-value problems for the stiffness ratios considered in the present paper reveals a number of phenomena that do not arise in the plane theory of elasticity. For example, the limiting equation is quasielliptic (has different orders of differentiation with respect to different variables), whereas in the plane theory of elasticity it is of composite type; for a rectangular region, the mixed problem reduces, in the limit, to two bending equations for an elastic beam. The problems examined in the present paper arise, for example, in studies of elastic plates reinforced in one direction by one family of very stiff continuous fibers.

1. We assume that the Kirchhoff–Love hypotheses are valid and the moments are related to the strains by

$$M_{11} = -(D_{11}e_{11}(w) + D_{12}e_{22}(w)), \qquad M_{22} = -(D_{12}e_{11}(w) + D_{22}e_{22}(w)),$$
$$M_{12} = -2D_{66}e_{12}(w). \tag{1.1}$$

The requirement of positive potential strain energy leads to the inequalities

$$D_{ii} > 0 \quad (i = 1,\ 2,\ 6), \quad D_{11}D_{22} - D_{66}^2 > 0.$$

We assume that the stiffness in one of the chosen directions far exceeds the other stiffnesses:

$$D_{11} \gg D_{22}, D_{12}, D_{66}.$$

We set

$$\varepsilon^{-2} = D_{11}/D_{22}, \quad d_{ij} = D_{ij}/D_{22}, \quad i, j = 1,\ 2,\ 6, \quad m = d_{12} + 2d_{66}, \quad \varepsilon \ll 1,$$
$$e_{11}(w) = \cos^2\alpha w_{,11} + \sin^2\alpha w_{,22} + \sin 2\alpha w_{,12},$$
$$e_{22}(w) = \cos^2\alpha w_{,22} + \sin^2\alpha w_{,11} - \sin 2\alpha w_{,12},$$
$$e_{12}(w) = \sin\alpha\cos\alpha(w_{,11} - w_{,22}) - \cos 2\alpha w_{,12}.$$

Here α is the angle between the above-mentioned direction and the x_1 axis, w is the deflection, and $w_{,ij} = \partial^2 w/\partial x_i \partial x_j$ ($i, j = 1, 2$). It should be noted that for rectilinear anisotropy, relations (1.1) coincide with the generally accepted relations; relations (1.1) imply that the orthotropy axes of the material make angles α and $\alpha + \pi/2$ with the x_1 axis. The problems considered reduce to the problem of minimization of the energy functional

$$\int_Q [D_{11}e_{11}^2(w) + D_{22}e_{22}^2(w) + 2D_{12}e_{11}^2(w) + 4D_{66}e_{12}^2(w)]\,dx - 2\int_Q fw\,dx, \quad f \in L^2(Q)$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 5, pp. 195–201, September–October, 1999. Original article submitted April 2, 1996; revision submitted January 19, 1998.

0021-8944/99/4005-0951 $22.00

in a closed subspace $H^2(Q)$, where Q is a plane bounded region with a piecewise-smooth boundary. The deflection is determined from the equation

$$D_{11}e_{11}^{*}e_{11}(w^{\varepsilon}) + D_{22}e_{22}^{*}e_{22}(w^{\varepsilon}) + D_{12}e_{11}^{*}e_{22}(w^{\varepsilon}) + D_{12}e_{22}^{*}e_{11}(w^{\varepsilon}) + 2D_{66}e_{12}^{*}e_{12}(w^{\varepsilon}) = f. \qquad (1.2)$$

Here e_{ij}^{*} are differential operators that are formally conjugate after Lagrange to the differential operators $e_{ij}(w^{\varepsilon})$ $(i, j = 1, 2)$. The superscript ε at the deflection indicates that the solution depends on a small parameter. Dividing the coefficients and the right side by D_{22} and retaining the previous notation for the dimensionless right side, we obtain the following equation with the large parameter ε^{-2} at the expression $e_{11}^{*}e_{11}(w^{\varepsilon})$:

$$\varepsilon^{-2}e_{11}^{*}e_{11}(w^{\varepsilon}) + d_{22}e_{22}^{*}e_{22}(w^{\varepsilon}) + d_{12}e_{11}^{*}e_{22}(w^{\varepsilon}) + d_{12}e_{22}^{*}e_{11}(w^{\varepsilon}) + 2d_{66}e_{12}^{*}e_{12}(w^{\varepsilon}) = f. \qquad (1.3)$$

Before studying the dependence of the solution of Eq. (1.3) on the small parameter, we consider the equation $e_{11}(v) = 0$. Obviously, this is a parabolic equation with the double family of real characteristics

$$\cos\alpha\psi_{,x_1} + \sin\alpha\psi_{,x_2} = 0.$$

We assume that the family of characteristics is sufficiently smooth and introduce the new coordinates $\xi = x_1$ and $\eta = x_2 - \psi(x_1)$. Then $e_{11}(v)$ is written as

$$e_{11}(v) = \frac{1}{1+\psi_{,\xi}^2}v_{,\xi\xi} - \frac{\psi_{,\xi\xi}}{1+\psi_{,\xi}^2}v_{,\eta} = a(\xi)v_{,\xi\xi} - b(\xi)v_{,\eta}.$$

We formulate the first boundary-value problem for Eq. (1.3):

$$w^{\varepsilon}\Big|_{\partial Q} = \psi_1, \qquad \frac{\partial w^{\varepsilon}}{\partial n}\Big|_{\partial Q} = \psi_2. \qquad (1.4)$$

Here ∂Q is the boundary of the region Q. We consider the question of solvability of the boundary-value problem (1.3), (1.4). Without loss of generality, the boundary conditions can be considered homogeneous; indeed, since the problem is linear, the solution w^{ε} can be written as the sum $w^{\varepsilon} = u + v$, where the function v satisfies the inhomogeneous boundary conditions and the function u is unknown. Generally, this leads to the appearance of a term of order $O(\varepsilon^{-2})$ on the right side of the equation for u. We first consider the case of the homogeneous boundary conditions (1.4). We show that Eq. (1.3) is uniformly elliptic. When $\alpha = 0$, the coefficients are constant and the existence of a unique solution of the problem (1.3), (1.4) is well known (see, e.g., [2, Chap. 4, Theorem 1.2]). For arbitrary α, it suffices to use the identity

$$e_{11}^2(w) + e_{22}^2(w) + 2e_{12}^2(w) = w_{,11}^2 + w_{,22}^2 + 2w_{,12}^2,$$

whose validity is proved by direct calculations, and the positive definiteness of the specific strain energy. Let $a^{\varepsilon}(w, v)$ be the bilinear symmetric form

$$a^{\varepsilon}(w,v) = \int_Q [\varepsilon^{-2}e_{11}(w)e_{11}(v) + d_{12}e_{11}(w)e_{22}(v)$$

$$+d_{12}e_{11}(v)e_{22}(w) + 4d_{66}e_{12}(w)e_{12}(v) + e_{22}(w)e_{22}(v)]\,dx.$$

The variational formulation of the problem (1.3), (1.4) consists of the following: to determine a function $w^{\varepsilon} \in H_0^2(Q)$ such that for any $v \in H_0^2(Q)$, the integral identity

$$a^{\varepsilon}(w^{\varepsilon}, v) = (f, v), \qquad f \in L^2(Q) \qquad (1.5)$$

holds. For the solution of the problem (1.5), the following estimates are valid:

$$\|w^{\varepsilon}\|_2 \leqslant C, \qquad \varepsilon^{-2}\|e_{11}(w^{\varepsilon})\|_0^2 \leqslant C. \qquad (1.6)$$

Indeed, there exists a positive number d such that $dd_{22} - d_{12}^2 > 0$. This condition and the compactness of embedding of $H_0^2(Q)$ in $L^2(Q)$ leads to the first estimate in (1.6). Choosing a small ε such that $\varepsilon^{-2} > d$, we obtain the second estimate. Therefore, for $\varepsilon \to +0$, we can determine a subsequence (for which we use the previous notation) that weakly converges to a certain function $w^0 \in H_0^2(Q)$ with $e_{11}(w^0) = 0$. We consider two versions: (a) $b(\xi) \neq 0$; (b) $b(\xi) = 0$.

We proved in Sec. 3 that a homogeneous boundary-value problem for the equation $e_{11}^* e_{11}(w^0) = 0$ has only a zero solution. Therefore, the solution of the boundary-value problem with homogeneous boundary values converges to zero in the limit $\varepsilon \to +0$ if the assumptions (a) and (b) are valid. In Sec. 3, we also studied the behavior of the inhomogeneous boundary-value problem as the small parameter tends to zero.

Theorem 1. *The solution of the boundary-value problem* (1.3), (1.4) *converges to zero as* $\varepsilon \to +0$ *if one of the assumptions* (a) *or* (b) *holds.*

It should be noted that the estimates (1.6) are valid not only in $H_0^2(Q)$ but also in $H^2(Q)$, and hence the subspace K specified by the condition $e_{11}(v) = 0$ can be nontrivial. Let us consider the following example. We assume that $\alpha = 0$ (rectilinear anisotropy) and Q is a rectangular region ($Q = \{(x_1, x_2); |x_1| \leqslant h, 0 \leqslant x_2 \leqslant 1\}$) and consider the following mixed boundary-value problem for the bending D_ε of an orthotropic plate:

$$w^\varepsilon(\pm h, x_2) = 0, \quad \frac{\partial w^\varepsilon}{\partial x_2}(\pm h, x_2) = 0, \quad \Big(\varepsilon^{-2} \frac{\partial^2 w^\varepsilon}{\partial x_1^2} + d_{12} \frac{\partial^2 w^\varepsilon}{\partial x_2^2}\Big)(x_1, 0) = 0,$$

$$\Big(\varepsilon^{-2} \frac{\partial^2 w^\varepsilon}{\partial x_1^2} + d_{12} \frac{\partial^2 w^\varepsilon}{\partial x_2^2}\Big)(x_1, 1) = 0, \quad \Big(\varepsilon^{-2} \frac{\partial^3 w^\varepsilon}{\partial x_1^3} + (d_{12} + 2d_{66}) \frac{\partial^3 w^\varepsilon}{\partial x_1 \partial x_2^2}\Big)(x_1, 0) = 0,$$

$$\Big(\varepsilon^{-2} \frac{\partial^3 w^\varepsilon}{\partial x_1^3} + (d_{12} + 2d_{66}) \frac{\partial^3 w^\varepsilon}{\partial x_1 \partial x_2^2}\Big)(x_1, 1) = 0.$$

Let V be the subspace of $H^2(Q)$ specified by the conditions

$$V = \{v \in H^2(Q), \partial^k v / \partial x_2^k = 0, \quad k = 1, 2\}$$

(the function and its derivative vanish for $x_2 = 0$ in a weak sense). The estimates (1.6) are valid, and, hence, from the sequence w^ε, we can distinguish a subsequence that is weakly converging to the element w^0 in V, such that $w^0_{,x_1x_1} = 0$. In this case, w^0 is written as the sum $w^0 = x_1\psi_1(x_2) + \psi_2(x_2)$. We consider the integral identity (1.5) in the subspace K of the space V specified by the condition $v_{,x_1x_1} = 0$ and pass to the limit $\varepsilon \to +0$. For w^0, we obtain the integral identity

$$\int_Q \{4d_{66} w^0_{,x_1x_2} \varphi_{,x_1x_2} + w^0_{,x_2x_2} \varphi_{,x_2x_2}\}\, dx = \int_Q f\varphi\, dx, \tag{1.7}$$

which is valid for any function $\varphi \in K$. Any function from K admits the representation $\varphi = x_1\varphi_1(x_2) + \varphi_2(x_2)$. In this case, w^ε and $\partial w^\varepsilon / \partial x_2$ converge, respectively, to w^0 and $\partial w^0/\partial x_2$ in $H^{3/2}(\partial Q)$ and $H^{1/2}(\partial Q)$, respectively and, hence, $\psi_k \in H_0^2(0,1)$, where $k = 1, 2$. We substitute ψ_k and ψ_1 into (1.7) and integrate. As a result, ψ_1 and ψ_2 and satisfy the integral identity

$$(2h^3/3) \int_0^1 \psi_{1,22}\varphi_{1,22}\, dx_2 + 2h \int_0^1 \psi_{2,22}\varphi_{2,22}\, dx_2$$

$$+4hd_{66} \int_0^1 \psi_{1,2}\varphi_{1,2}\, dx_2 = \int_0^1 \langle x_1 f\rangle \varphi_1(x_2)\, dx_2 + 2h \int_0^1 \langle f\rangle\, dx_2 \tag{1.8}$$

for any $\varphi_k(x_2) \in H_0^2(0,1)$. Here

$$\langle g\rangle = \int_{-h}^{h} g(x_1, x_2)\, dx_1.$$

Since $\langle g\rangle \in L^2(0,1)$, in accordance with the well-known smoothness theorems, we have ψ_1 and $\psi_2 \in H^4(0,1)$ and, hence, they are solutions of the equations

$$\frac{2h^3}{3} \frac{d^4\psi_1}{dx_2^4} - 4hd_{66} \frac{d^2\psi_1}{dx_2^2} = \langle fx_1\rangle, \qquad \frac{d^4\psi_2}{dx_2^4} = \langle f\rangle. \tag{1.9}$$

The first equation in (1.9) can be regarded as a bending equation for a prestressed beam. Thus, the limiting problem is split into two problems of beam bending.

Theorem 2. *Under the above assumptions, the solution of the problem D_ε converges in V to the solution of the problem* (1.9).

2. In this situation, the behavior of the solution of the problem of plate bending above an obstacle is of interest. Let K_0 be a cone in V specified by the condition $v \geqslant 0$ on ∂Q. We consider the variational inequality

$$a^\varepsilon(w^\varepsilon, v - w^\varepsilon) \geqslant (f, v - w^\varepsilon), \qquad v \in K_0. \tag{2.1}$$

Direct passage to the limit in (2.1) is impossible since the left side in (2.1) contains a negative power of ε. The existence of solutions of the problem (2.1) is well known and it follows from the general theorems of the theory of monotonic operators [3]. Instead of (2.1) we consider the problem with the penalty operator

$$a^\varepsilon(w^{\varepsilon,\eta}, v) - \eta^{-1} \int\limits_Q (w^{\varepsilon,\eta})^- v \, dx = (f, v), \quad f \in L^2(Q). \tag{2.2}$$

Here

$$v^-(x) = 0, \quad v(x) \geqslant 0, \quad v^-(x) = -v(x), \quad v(x) < 0.$$

For the solution of the problem (2.2), the following estimates, which are uniform in ε and η, are valid:

$$\|w^{\varepsilon,\eta}\|_2 \leqslant C, \qquad \varepsilon^{-2} \|w^{\varepsilon,\eta}_{,x_1x_1}\|_0^2 \leqslant C.$$

Using these estimates, we pass in (2.2) to the limit $\varepsilon \to +0$. In this case, the term with the penalty operator converges strongly in $L^2(Q)$. In the limit, we obtain

$$w^{0,\eta} = x_1 \psi_1^\eta(x_2) + \psi_2^\eta(x_2).$$

We introduce the symmetric bilinear forms

$$a^1(\psi_1^\eta, \varphi_1) = \frac{2h^3}{12} \int\limits_0^1 \psi^\eta_{,22} \varphi_{1,22} \, dx_2 + 2hd_{66} \int\limits_0^1 \psi^\eta_{1,2} \varphi_{1,2} \, dx_2,$$

$$a^2(\psi_2^\eta, \varphi_2) = \int\limits_0^1 \psi^\eta_{2,22} \varphi_{2,22} \, dx_2.$$

As a result, ψ_1^η and ψ_2^η satisfy the integral identities

$$a^1(\psi_1^\eta, \varphi_1) - \frac{2h^3}{3} \eta^{-1} \int\limits_0^1 (\psi_1^\eta)^- \varphi_1(x_2) \, dx_2 = \int\limits_0^1 (\langle x_1 f \rangle + 2h \langle f \rangle) \varphi_1(x_2) \, dx_2; \tag{2.3}$$

$$a^2(\psi^\eta, \varphi_2) - \eta^{-1} \int\limits_0^1 (\psi_2^\eta)^- \varphi_2(x_2) \, dx_2 = \int\limits_0^1 \langle f \rangle \varphi_2(x_2) \, dx_2. \tag{2.4}$$

The integral identities (2.3) and (2.4) correspond to the penalty equations for the problem of beam bending above an obstacle. Passage to the limit $\eta \to +0$ is performed by the well-known procedure [3]. We denote by ψ_1^0 and ψ_2^0 the weak limits of the functions ψ_1^η and ψ_2^η as $\eta \to +0$. Hence, ψ_1^0 and ψ_2^0 satisfy the variational inequalities

$$a^1(\psi_1^0, \varphi_1 - \psi_1^0) \geqslant (\langle f x_1 \rangle, \varphi_1 - \psi_1^0) \ \ \forall \varphi_1 \in K_0 \cap H_0^2(0,1); \tag{2.5}$$

$$a^2(\psi_2^0, \varphi_2 - \psi_2^0) \geqslant (\langle f \rangle, \varphi_2 - \psi_2^0) \ \ \forall \varphi_2 \in K_0 \cap H_0^2(0,1). \tag{2.6}$$

Thus, in the limit, the initial variational inequality (2.1) is split into two inequalities corresponding to the problem of bending of an elastic beam above an obstacle.

Theorem 3. *In V, the solution of the variational inequality* (2.1) *converges weakly to the solutions of the inequalities* (2.5) *and* (2.6).

3. In Sec. 1, we showed that the solution of the problem (1.2) converges to zero as $\varepsilon \to +0$. If the boundary data are nonzero, in reducing the boundary-value problem to the homogeneous problem, we obtain a term of order $O(\varepsilon^{-2})$ on the right side of the equation. We investigate the behavior of the solution of the problem

$$a^\varepsilon(w^\varepsilon, v) = (\varepsilon^{-2} f, v), \quad f \in L^2(Q), \quad w^\varepsilon \in \overset{\circ}{H}{}^2_0(Q) \tag{3.1}$$

as $\varepsilon \to +0$. We recall that changing the independent variables under the assumption of sufficient smoothness of the coefficients, we can bring the differential operator $e_{11}(w)$ to the form

$$e_{11}(w) = (1 + \varphi_{,x_1}^2)^{-1} w_{,x_1x_1} - \varphi_{,x_1x_1}(1 + \varphi_{,x_1}^2)^{-2} w_{,x_2}.$$

For brevity, we set $L(w) = e_{11}(w)$. In the sequel, it is necessary to examine the solvability of the first boundary-value problem for the equation

$$L^*(L(w)) = f. \tag{3.2}$$

We note that the smoothness properties of the solution of Eq. (3.2) depend heavily on whether the function $g(x_1) = \varphi_{,x_1x_1}$ vanishes everywhere in the region or not. If $g(x_1) = 0$, we define the space G_1 as a replenishment of the class of functions $C_0^\infty(Q)$ by the norm

$$\|u\|_{G_1} = \left\{ \int_Q [u_{,x_1x_1}^2 + u_{,x_1}^2 + u^2]\, dx \right\}^{1/2}. \tag{3.3}$$

Exact theorems on the traces of the functions from G_1 for a region with a piecewise-smooth boundary are given in [4]. If $g(x_1)$ is nonzero, we introduce the space $G_2 = W^{2,1}(Q)$ as a replenishment of functions of class $C_0^\infty(Q)$ by the norm

$$\|u\|_{G_2} = \left\{ \int_Q [u_{,x_1x_1}^2 + u_{,x_1}^2 + u_{,x_2}^2 + u^2]\, dx \right\}^{1/2}.$$

We denote by G the space G_1 in the first case and the space G_2 in the second case. In both cases, by the generalized solution of the first boundary-value problem for Eq. (3.2) is meant the function $w \in G$ satisfying the integral identity

$$\int_Q L(w)L(v)\, dx = \int_Q fv\, dx \tag{3.4}$$

for any $v \in C_0^\infty(Q)$. The boundary conditions for Eq. (3.2) have the form

$$w\Big|_{\partial Q^*} = 0, \qquad \frac{\partial w}{\partial x_1}\Big|_{\partial Q^*} = 0 \tag{3.5}$$

if $g(x_1) = 0$, and

$$w\Big|_{\partial Q} = 0, \qquad \frac{\partial w}{\partial x_2}\Big|_{\Gamma_2} = 0, \qquad \frac{\partial w}{\partial n}\Big|_{\Gamma^*} = 0, \qquad \Gamma^* = \partial Q - \Gamma_1 \cup \Gamma_2 \tag{3.6}$$

if $g(x_1)$ is nonzero. Here ∂Q^* is the noncharacteristic part of the boundary and Γ_i ($i = 1$ and 2 and $\operatorname{mes}\Gamma_i \neq 0$) are the boundary segments described by the equation $x_i = \text{const}$. For the functions from G, an inequality of the Poincaré type is valid: there exists a positive constant C such that the inequality

$$\|u\|_0^2 \leqslant C \left\| \frac{\partial u}{\partial x_i} \right\|_0^2 \tag{3.7}$$

holds for any function from G. Here $i = 1$ if $u \in G_1$ and $i = 1$ and 2 if $u \in G_2$. From the last inequality it follows that if $u \in G_1$, the seminorm $\left| u_{,x_1x_1} \right|_0^2$ [the norm of the function $u_{,x_1x_1}$ in $L^2(Q)$] on G_1 is equivalent to the norm (3.3) and, hence, the boundary-value problem for Eq. (3.2) with boundary conditions (3.5) has a unique solution. When $g(x_1)$ is nonzero, the situation is somewhat more complicated. Mikhailov [5] proved (Lemma 8.1) that, for the generalized solution of the problem (3.2) with boundary conditions (3.6), an analog of the Gårding inequality for elliptic operators is valid; i.e., there exist positive constants C_1 and C_2 such that

$$(L(v), L(v)) \geqslant C_1\|v\|_{G_2}^2 - C_2\|v\|_0^2. \tag{3.8}$$

Let us show that the problem (3.2), (3.6) has a unique solution in G_2. Indeed, if f is equal to zero, the integral identity implies that $L(w) = 0$. In other words, we have

$$\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial x_1^2}\frac{\partial w}{\partial x_2} = 0. \tag{3.9}$$

We multiply (3.9) by w and integrate the result by parts. We thus infer that $\|w_{x_1}\|_0 = 0$. Since the Poincaré-type inequality (3.7) is valid for w, it follows that $w = 0$. From the uniqueness of the solution and inequality (3.8), it follows that there exists a positive constant K such that the inequality

$$(L(w), L(w)) \geqslant K\|w\|_{G_2}^2 \tag{3.10}$$

is valid. Indeed, if inequality (3.10) fails, for each natural number n, we can find an element $u_n \in G_2$ such that $\|u_n\|_{G_2}^2 \geqslant n(L(u_n), L(u_n))$. We set $v_n = u_n/\|u_n\|_{G_2}$. Then, $\|v_n\|_{G_2} = 1$ for each n and $(Lv_n, Lv_n)_0 \leqslant 1/n$. Consequently, $(Lv_n, Lv_n)_0 \to 0$ as $n \to \infty$. Therefore, $(Lv_n, Lv)_0$ tends to zero as $n \to \infty$ for each $v \in G_2$; but because of the compactness of embedding of $G_2(Q)$ in $L^2(Q)$, there exists a subsequence (for which we use the previous notation) such that $\|u_j - u_k\|_0$ tends to zero as j, $k \to \infty$. However, by virtue of inequality (3.7), we have

$$(Lu_j - Lu_k, Lu_j - Lu_k)_0 \geqslant C_1\|u_j - u_k\|_{G_2}^2 - C_2\|u_j - u_k\|_0^2,$$

and, hence, $\|u_j - u_k\|_{G_2} \to 0$ and there exists (in view of the completeness) a function u such that $\|u_j - u\|_{G_2} \to 0$. Then, $(Lu, Lv) = 0$ for any $v \in G_2$. But the unique solution in G_2 is zero. We arrive at a contradiction since the norm of u in G_2 is equal to unity. We revert to the problem (3.1).

Theorem 4. *In G, the solution of the problem* (3.1) *converges weakly to the solution of the problem* (3.8), *and the following estimates are valid:*

$$\|w^\varepsilon\|_G \leqslant C, \qquad \varepsilon^2\|e_{k2}(w^\varepsilon)\|_0^2 \leqslant C \quad (k = 1, 2) \tag{3.11}$$

Here C is independent of ε. We multiply the integral identity (3.8) by ε^2 and set $v = w^\varepsilon$. We thus obtain the estimate

$$\|e_{11}(w^\varepsilon)\|_0^2 + C_1\varepsilon^2\|e_{12}(w^\varepsilon)\|_0^2 + C_2\varepsilon^2\|e_{22}(w^\varepsilon)\|_0^2 \leqslant \|f\|_0\|w^\varepsilon\|_0 \leqslant (4\theta)^{-1}\|f\|_0^2 + \theta\|w^\varepsilon\|_G^2. \tag{3.12}$$

Since $\|e_{11}(w^\varepsilon)\|_0^2 \geqslant C\|w^\varepsilon\|_G^2$, setting $\theta = C/2$ in (3.12), we obtain the estimate $\|w^\varepsilon\|_G \leqslant C\|f\|_0$ and, hence, the other estimates in (3.12). Estimates (3.12) make it possible to pass to the limit in the integral identity (1.5) multiplied by ε^2; here w^ε converges weakly to w^0 in G, and w^0 satisfies the integral identity (3.9). From the uniqueness of the solution, it follows that the entire sequence converges to w^0.

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